An RMS Particle Core Model for Rings

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ABSTRACT

A self-consistent set of equations for the azimuthal variation of rms betatron oscillation amplitudes, including the effects of dispersion and space charge, is derived. These effective envelope equations can be integrated over the beam energy distribution to provide space charge forces in a particle core model for rings. The derivation of the envelope equations involves an accelerator ordering scheme for the beam dynamics and a statistical moments analysis of the canonical distribution function in the six-dimensional phase space of the beam Hamiltonian. The azimuthal variation of the second moments of the transverse canonical coordinates, x^b and z^b, integrated over the kinetic distribution function of the beam, provides the rms equations. These equations, at fixed beam energy, are integrated over the beam energy distribution to provide the overall space charge distribution and force. Because the envelope equations and dispersion function both depend upon and determine the space charge forces, the consistency of the core model requires either analytic closure assumptions or numerical iteration.

INTRODUCTION

The analysis and understanding of space charge effects for particle beams in linear accelerators has been greatly facilitated by the use of particle core models (1). Particle core models represent the dynamics of the beam by envelope equations that contain the effects of the lattice focusing forces, the beam emittances, and the space charge. In addition to providing a collective model for beam dynamics, the envelope equations are also used to calculate space charge forces in particle tracking. Particle core models are employed in conjunction with particle-tracking calculations to simultaneously study both collective and individual particle dynamics.

The alternatives to particle core models for the computational study of space charge effects are the Particle in Cell (PIC) models (2). In comparison with PIC models, particle core models have both advantages and disadvantages. One advantage of particle core models involves computing time. The computational work involved in advancing the envelope equations is comparable to that in tracking a single particle, and the evaluation of the resulting space charge forces is fast. Another advantage of particle core models is their simplicity, making them amenable to analytic calculations. A third advantage is that individual particle orbits can be studied one at a time using space charge fields given by the particle core model. However, a major limitation of the particle core model is that it provides a simplified representation of the space charge distribution, both in the envelope equations and in the particle-tracking equations. Even so, particle core models provide a practical middle ground between analytic theory and large PIC code calculations.

Up until now, particle core models have been applied primarily to the study of beam dynamics and halo generation in straight channels with strong space charge forces. Now, with the advent of a number of applications involving rings with high beam intensities and small beam loss requirements (3), the application of particle core models to rings is necessary. However, the representation of the beam via the envelope equations must be generalized to include the effects of dispersion. One treatment of this dispersion problem has recently been carried out at University of Maryland (4), and we present a somewhat different approach here.

The purpose of this paper is to extend the particle core model to rings by including dispersion. The approach involves a moments analysis of the betatron oscillations averaged over the distribution function in the canonical phase space of a beam Hamiltonian which is derived using an ordering scheme that separates the betatron motion from the longitudinal motion and the dispersion. Effective envelope equations are derived for the azimuthal variation of the rms values of the canonical transverse phase space coordinates, with the statistical averaging at each energy over the beam distribution function in phase space. These envelope equations are independent of dispersion and valid, in principle, for arbitrary space charge distributions. The effects of dispersion are incorporated in a straightforward fashion by integrating the envelope equations over beam energy to obtain the overall space charge distribution. For self-consistency, this space charge distribution must be the same as that appearing in the dispersion and envelope equations. In subsequent sections we present the accelerator ordering scheme and the beam Hamiltonian derivation; the behavior of the distribution function and integrated quantities under the accelerator ordering scheme; the derivation of the rms envelope equations; and the incorporation of dispersion to obtain the overall charge distribution.

THE BEAM HAMILTONIAN

This section presents a derivation of the beam Hamiltonian in a form that decouples the transverse betatron motion from the longitudinal motion and from explicit dependence on the dispersion. Although a derivation was recently published by one of the authors (5), the present version uses an accelerator ordering scheme, which is later applied to derive the envelope equations. The main purposes of presenting this section here are 1) to introduce the ordering scheme, and 2) to obtain the canonical variables of the phase space used later in deriving envelope equations.

Begin with the standard Hamiltonian for a charged particle in an electromagnetic field:

$$\begin{aligned} H &= q\Phi + [m^2c^4 + (c\vec{p} - q\vec{A})^2]^{\frac{1}{2}} \\ &= E_0 + \Delta E, \end{aligned} \tag{1}$$

where $E_0 \equiv gmc^2$ is the reference energy of the beam; ΔE is the energy deviation; the canonical coordinates and momenta are (x, p_x, z, p_z, s, p_s); and the time, t, is the independent parameter. In order to make the azimuthal coordinate, s, the independent parameter, it is customary (6) to define a new Hamiltonian by $H_1 = -p_s$. First, separate the vector potential $\vec{A}$ into contributions $\vec{A}^{ext}$ from the lattice and $\vec{A}^{sc}$ from the beam space charge. We assume the lattice contributions to the magnetic fields are transverse to the azimuthal direction, so that the external vector potential can be chosen to satisfy $A_x^{ext} = A_z^{ext} = 0$. Define a reference momentum that satisfies the equation $p_0^2 c^2 \equiv E_0^2 - m^2 c^4 = b^2 E_0^2$. With these assumptions and definitions, Eq. (1) can be manipulated to obtain:

$$
\begin{aligned}
H_1 = & -\frac{qA_s^{ext}}{c} - \frac{qA_s^{sc}}{c} - (1 + \frac{x}{r}) \times \\
& [p_0{}^2 + 2\frac{E_0}{c^2}(\Delta E - q\Phi) + \frac{(\Delta E - q\Phi)^2}{c^2} - (p_x - \frac{qA_x^{sc}}{c})^2 - (p_z - \frac{qA_z^{sc}}{c})^2]^{\frac{1}{2}}.
\end{aligned}
\tag{2}
$$

The canonical variables are now $(x, p_x, z, p_z, t, -\Delta E)$. To adapt Eq. (2) to the analysis of space charge in rings, we adopt the following accelerator ordering scheme:

$$
\begin{aligned}
&O(e^{-1}) && - \frac{\P}{\P x}, \frac{\P}{\P z}, \frac{\P}{\P p_x}, \frac{\P}{\P p_z}, \frac{\P}{\P(\Delta E)} \\
&O(1) && - H_1, r, H, mc^2, E_0, t, s, \frac{\P}{\P t}, \frac{\P}{\P s} \\
&O(e) && - \vec{A}^{ext}, x, p_x, z, p_z, \Delta E \\
&O(e^2) && - \vec{A}^{sc}, \Phi.
\end{aligned}
$$

The application of this ordering to Eq. (2) yields, valid through $O(e^2)$:

$$
\begin{aligned}
H_1 = & -p_0 - \frac{x}{r} p_0 - \frac{qA_s^{ext}}{c} + \frac{p_0}{2}[(\frac{p_x}{p_0})^2 + (\frac{p_z}{p_0})^2] \\
& - \frac{E_0}{p_0 c^2}\Delta E - \frac{x}{r}\frac{E_0}{p_0 c^2}\Delta E + \frac{1}{2p_0 c^2}\frac{1}{g^2 b^2}(\Delta E)^2 + q(\frac{E_0}{p_0 c^2}\Phi - \frac{A_s^{sc}}{c}).
\end{aligned}
\tag{3}
$$

The external vector potential is next expressed in terms of the magnet fields and the space charge vector potential in terms of the electrostatic potential. We will not consider accelerating channels here, but rather confine our attention to nonaccelerated beams, so that A_s^{ext} is independent of time. An expression for the external potential, valid through second order, for bending and quadrupole magnets is $A_s^{ext} = -B_0 x - \frac{1}{2}(\frac{B_0}{\rho} + \frac{\partial B_0}{\partial x})x^2 + \frac{1}{2}\frac{\partial B_0}{\partial x} z^2$, where B_0 is the bending field on the reference orbit. The vector potential due to the beam is related to the electrostatic potential by $A_s^{sc} = \beta\Phi$. Together with these substitutions, we make the following canonical scaling transformation:

$$
\begin{aligned}
&H_2 = \frac{H_1}{p_0} \\
&\tilde{p}_x = \frac{p_x}{p_0} \qquad \tilde{p}_z = \frac{p_z}{p_0} \qquad \tilde{\delta} = \frac{\Delta E}{\beta^2 E_0} = \frac{dp}{p_0} \\
&\tilde{x} = x \qquad \tilde{z} = z \qquad \tilde{t} = \beta ct\,.
\end{aligned}
\tag{4}
$$

With the above transformation and substitutions, and using $\frac{qB_0}{p_0 c} = \frac{1}{\rho}$, we obtain:

$$
\begin{aligned}
H_2(\tilde{x}, \tilde{p}_x, \tilde{z}, \tilde{p}_z, \tilde{t}, -\tilde{\delta}, s) = {} & \frac{1}{2}[(\tilde{p}_x)^2 + K_x(s)(\tilde{x})^2 + (\tilde{p}_z)^2 + K_z(s)(\tilde{z})^2] \\
& -\tilde{\delta} - \frac{\tilde{x}}{\rho}\tilde{\delta} + \frac{1}{2\gamma^2}(\tilde{\delta})^2 + \frac{q}{\gamma^2\beta^2 E_0}\Phi(\tilde{x}, \tilde{z}, s),
\end{aligned}
\tag{5}
$$

where $K_x(s) = \frac{1}{\rho^2}(1 + \frac{\rho}{B_0}\frac{\partial B_0}{\partial x})$; $K_z(s) = \frac{1}{\rho B_0}\frac{\partial B_0}{\partial x}$; and a term of constant value -1 was dropped. The canonical variables are now $(\tilde{x}, \tilde{p}_x, \tilde{z}, \tilde{p}_z, \tilde{t}, -\tilde{\delta})$ and the independent parameter is s.

Equation (5) contains the term $-\frac{\tilde{x}}{\rho}\tilde{\delta}$, which couples the betatron motion to the dispersion. The definitive step in Ref. (5) removes this term through the following canonical transformation of the second type:

$$
F_2(\tilde{x}, p_x^b, \tilde{z}, p_z^b, \tilde{t}, -\delta, s) = (\tilde{x} - D_x\delta)p_x^b - \tilde{t}\,\delta + \tilde{z}\,p_z^b + D_x{}'\,\tilde{x}\delta - \frac{1}{2}D_x D_x{}'\delta^2\,, \tag{6}
$$

where $D_x = D_x(s)$ is a function of s that is yet to be defined. The quantities D_x' and D_x'' are the first and second derivatives of D_x with respect to s. With this transformation the new coordinates and momenta are

$$\begin{aligned} x^b &= \tilde{x} - D_x d \qquad & z^b &= \tilde{z} \qquad & d &= \tilde{d} \\ p_x^b &= \tilde{p}_x - D_x' d \qquad & p_z^b &= \tilde{p}_z \qquad & t &= \tilde{t} + D_x p_x - D_x' x, \end{aligned} \tag{7}$$

and the new Hamiltonian is

$$\begin{aligned} H(x^b, p_x^b, z^b, p_z^b, t, -d, s) = \frac{1}{2}[(p_x^b)^2 + K_x(s)(x^b)^2 + (p_z^b)^2 + K_z(s)(z^b)^2] \\ -d - \frac{1}{2}d^2[\frac{D_x(s)}{r} - \frac{1}{g^2}] \\ +x^b d[D_x''(s) + K_x(s)D_x(s) - \frac{1}{r}] \\ +\frac{1}{2}d^2 D_x(s)[D_x''(s) + K_x(s)D_x(s) - \frac{1}{r}] \\ +\frac{q}{g^2 b^2 E_0}\Phi(x^b + D_x(s)d, z^b, s). \end{aligned} \tag{8}$$

Clearly, from Eqs. (7) and (8), D_x is closely related to the dispersion function, and the new transverse coordinates and momenta, x^b, p_x^b, y^b, p_y^b, are pure betatron oscillations about the closed orbit at the specific corresponding energy. Although d is the relative momentum deviation, we will use the terms momentum and energy interchangeably in referring to d here.

To complete the separation of the Hamiltonian, we consider the dependence of the space charge term on D_x. To do this, we expand $\Phi(x^b + D_x d, z^b, s)$ about the reference orbit:

$$\Phi(x^b + D_x d, z^b, s) = \sum_{m,n=0}^{\infty} \frac{\Phi^{mn}(s)}{m!n!}[x^b + D_x(s)d]^m (z^b)^n. \tag{9}$$

We assume that the beam is centered at the origin, so that $\Phi^{01} = \Phi^{10} = 0$, and is sufficiently symmetric that $\Phi^{11} = 0$. This is certainly true for most commonly used analytic beam distributions. Then, setting the constant term $\Phi^{00} = 0$, Eq. (9) can be written as

$$\begin{aligned}\Phi(x^b + D_x d, z^b, s) &= \frac{\Phi^{20}}{2}(x^b + D_x d)^2 + \frac{\Phi^{02}}{2}(z^b)^2 + \Delta\Phi \\ &= \frac{\Phi^{20}}{2}(x^b)^2 + \frac{\Phi^{02}}{2}(z^b)^2 + \Phi^{20} D_x x^b d + \frac{\Phi^{20}}{2}(D_x d)^2 + \Delta\Phi,\end{aligned} \tag{10}$$

where the term $\Delta\Phi = O(e^3)$. Substituting this into Eq. (8) yields

$$\begin{aligned}H(x^b, p_x^b, z^b, p_z^b, t, -d, s) = &\frac{1}{2}\{(p_x^b)^2 + [K_x(s) + \frac{q\Phi^{20}(s)}{g^2 b^2 E_0}](x^b)^2\} \\ &+\frac{1}{2}\{(p_z^b)^2 + [K_z(s) + \frac{q\Phi^{02}(s)}{g^2 b^2 E_0}](z^b)^2\} \\ &-d - \frac{1}{2}d^2[\frac{D_x(s)}{r} - \frac{1}{g^2}] \\ &+x^b d\{D_x''(s) + [K_x(s) + \frac{q\Phi^{20}(s)}{g^2 b^2 E_0}]D_x(s) - \frac{1}{r}\} \\ &+\frac{1}{2}d^2 D_x(s)\{D_x''(s) + [K_x(s) + \frac{q\Phi^{20}(s)}{g^2 b^2 E_0}]D_x(s) - \frac{1}{r}\} \\ &+\frac{q}{g^2 b^2 E_0}\Delta\Phi(x^b + D_x d, z^b, s).\end{aligned} \tag{11}$$

We now choose D_x to be the dispersion function with lowest order space charge correction, meaning that D_x satisfies the equation

$$D_x''(s) + [K_x(s) + \frac{q\Phi^{20}(s)}{g^2 b^2 E_0}]D_x(s) - \frac{1}{r} = 0\,. \tag{12}$$

With this choice of dispersion function, the Hamiltonian finally simplifies to

$$\begin{aligned}H(x^b, p_x^b, z^b, p_z^b, t, -d, s) = &\frac{1}{2}\{(p_x^b)^2 + [K_x(s) + \frac{q\Phi^{20}(s)}{g^2 b^2 E_0}](x^b)^2\} \\ &+\frac{1}{2}\{(p_z^b)^2 + [K_z(s) + \frac{q\Phi^{02}(s)}{g^2 b^2 E_0}](z^b)^2\} \\ &-d - \frac{1}{2}d^2[\frac{D_x(s)}{r} - \frac{1}{g^2}] \\ &+O(e^3).\end{aligned} \tag{13}$$

The canonical coordinates and momenta are $(x^{\beta}, p_x^{\beta}, z^{\beta}, p_z^{\beta}, \tau, -\delta)$. The motion for the Hamiltonian in Eq. (13) is completely separated into independent contributions for betatron oscillations, (x^{β}, p_x^{β}) and (z^{β}, p_z^{β}), and longitudinal dynamics, (δ, τ). The betatron terms are independent of dispersion and of the momentum shift, δ; while the coordinate, τ, is cyclic, so that δ is constant. The space charge terms provide tune shifts in the betatron oscillations (first two lines) and modify the dispersion function (Eq. (12)) appearing in the third line. In lowest order all space charge terms are expressed with coefficients obtained from expanding the potential about the reference orbit. The remaining effects of space charge are third order and higher, and will be neglected in determining the rms envelope equations. In terms of the present notation, the ordering scheme is

$$
\begin{array}{lll}
O(\varepsilon^{-1}) & \sim & \dfrac{\partial}{\partial x^{\beta}}, \dfrac{\partial}{\partial z^{\beta}}, \dfrac{\partial}{\partial p_x^{\beta}}, \dfrac{\partial}{\partial p_z^{\beta}}, \dfrac{\partial}{\partial(-\delta)} \\
O(1) & \sim & K_x, K_z, q\Phi^{20}, q\Phi^{02}, D_x, \rho, \gamma, \beta, E_0, \tau, s, \dfrac{\partial}{\partial \tau}, \dfrac{\partial}{\partial s} \\
O(\varepsilon) & \sim & x^{\beta}, p_x^{\beta}, z^{\beta}, p_z^{\beta}, -\delta \\
O(\varepsilon^{2}) & \sim & H + \delta .
\end{array}
$$

DISTRIBUTION FUNCTION AND AVERAGING

Let $f(x^{\beta}, p_x^{\beta}, z^{\beta}, p_z^{\beta}, \tau, -\delta, s)$ be the distribution function of the beam in the phase space of canonical coordinates. Define the phase space volume elements $dV = dx^{\beta} dp_x^{\beta} dz^{\beta} dp_z^{\beta} d\tau d(-\delta)$ and $dV_{\perp} = dx^{\beta} dp_x^{\beta} dz^{\beta} dp_z^{\beta}$. We take the normalization of f to be given by $\int f dx^{\beta} dp_x^{\beta} dz^{\beta} dp_z^{\beta} d\tau d(-\delta) = \int f dV = N$, where N is the number of particles in the beam. The distribution function f satisfies a kinetic equation of the form

$$
\frac{\partial f}{\partial s} + (x^{\beta})' \frac{\partial f}{\partial x^{\beta}} + (p_x^{\beta})' \frac{\partial f}{\partial p_x^{\beta}} + (z^{\beta})' \frac{\partial f}{\partial z^{\beta}} + (p_z^{\beta})' \frac{\partial f}{\partial p_z^{\beta}} + \tau' \frac{\partial f}{\partial \tau} + (-\delta)' \frac{\partial f}{\partial(-\delta)} = S , \quad (14)
$$

where the source term S accounts for non-Hamiltonian processes, such as gains due to injection and losses due to collisions with the foil or with other beam particles. We extend the adopted ordering scheme by assuming that the distribution function is $O(1)$ and that the non-Hamiltonian sources and sinks are $O(\varepsilon)$:

$$
\begin{array}{lll}
O(1) & \sim & f \\
O(\varepsilon) & \sim & S .
\end{array}
$$

Then, to lowest order the distribution function obeys the Vlasov Equation in canonical phase space.

Useful averages and moments are calculated by integrating the product of the quantity to be averaged times the distribution function over phase space at fixed values of azimuth, s, time, t, and momentum, d. According to Eqs. (4) and (7), $t = bct + D_x p_x^b - D_x' x^b$. Hence, to average at fixed time, multiply the function to be averaged by the Dirac delta function, $d(t - bct - D_x p_x^b + D_x' x^b)$. Similarly, to average at fixed momentum, d_0, multiply by $d(d - d_0)$. Using this prescription, we define the following quantities:

$$
\begin{aligned}
l(s,t,d_0) &= \int d(t - bct - D_x p_x^b + D_x' x^b) d(d - d_0) f dV \\
&= \int f(x^b, p_x^b, z^b, p_z^b, bct + D_x p_x^b - D_x' x^b, -d_0, s) dV_\perp \\
l_{tot}(s,t) &= \int d(t - bct - D_x p_x^b + D_x' x^b) f dV \\
&= \int f(x^b, p_x^b, z^b, p_z^b, bct + D_x p_x^b - D_x' x^b, -d, s) dV_\perp d(-d) \\
P(d) &= \frac{l(s,t,d)}{l_{tot}(s,t)}.
\end{aligned} \tag{15}
$$

In Eq. (15), $l(s,t,d_0)$ is the azimuthal particle density per unit momentum at d_0, $l_{tot}(s,t)$ is the overall azimuthal particle density, and $P(d)$ is the momentum probability distribution: note that $\int P(d) d(-d) = 1$. Furthermore, the averages of any function, $g(x^b, p_x^b, z^b, p_z^b, t, -d, s)$, are given by:

$$
\begin{aligned}
\langle g(s,t,d) \rangle &= \frac{1}{l(s,t,d)} \int [g \times f](x^b, p_x^b, z^b, p_z^b, bct + D_x p_x^b - D_x' x^b, -d, s) dV_\perp \\
\langle g(s,t) \rangle_{tot} &= \int \langle g(s,t,d) \rangle P(d) d(-d).
\end{aligned} \tag{16}
$$

In order to derive rms envelope equations, it is necessary to differentiate averaged quantities in the azimuthal direction. Because these are dynamic equations, the derivatives in s are accompanied by changes in time. From Eqs. (4), (5), (7), and (13) it is seen that

$$
\begin{aligned}
bct' &= \frac{\P H_2}{\P(-\tilde{d})} = 1 + d\,(\frac{D_x}{r} - \frac{1}{g^{\,2}}) + \frac{x^b}{r} \\
t\,' &= \frac{\P H}{\P(-d)} = 1 + d\,(\frac{D_x}{r} - \frac{1}{g^{\,2}}).
\end{aligned}
\tag{17}
$$

In the s differentiation of quantities averaged over transverse phase space, it is appropriate to evaluate bct' on the closed orbit, $x^b = 0$. In moving from s to $s^+ = s + \Delta s$, the quantity $t = bct + D_x(s)p_x^b - D_x{}'(s)x^b$ becomes $t^{\,+} = bct + [1 + d(\frac{D_x}{r} - \frac{1}{g^{\,2}})]\Delta s + D_x(s+\Delta s)p_x^b - D_x{}'(s+\Delta s)x^b$. Accordingly, define $\frac{\P t}{\P s} = 1 + d(\frac{D_x}{r} - \frac{1}{g^{\,2}}) + D_x{}'\,p_x^b - D_x{}''\,x^b = t\,' + D_x{}'\,p_x^b - D_x{}''\,x^b$.

Then

$$
\frac{d}{ds}[\,l\,\langle g(s,t,d)\rangle] = \lim_{\Delta s \to 0}\frac{1}{\Delta s}\begin{bmatrix} \int [g \times f\,](x^b, p_x^b, z^b, p_z^b, t^{\,+}, -d, s^+)dV_\perp \\ -\int [g \times f\,](x^b, p_x^b, z^b, p_z^b, t, -d, s)dV_\perp \end{bmatrix}, \tag{18}
$$

and differentiation inside the integral and application of the chain rule yields $\frac{d}{ds}[\,l\,\langle g(s,t,d)\rangle] = \int[(\frac{\P g}{\P t}\frac{\P t}{\P s} + \frac{\P g}{\P s}) \times f + g \times (\frac{\P f}{\P t}\frac{\P t}{\P s} + \frac{\P f}{\P s})]dV_\perp$. Using the Vlasov Equation to substitute for $\frac{\P f}{\P s}$, we get

$$
\frac{d}{ds}[\,l\,\langle g(s,t,d)\rangle] = l\left\langle \frac{\P g}{\P t}\frac{\P t}{\P s} + \frac{\P g}{\P s}\right\rangle - \int \begin{pmatrix} g \times [(x^b)'\frac{\P f}{\P x^b} + (p_x^b)'\frac{\P f}{\P p_x^b} + (z^b)'\frac{\P f}{\P z^b} + (p_z^b)'\frac{\P f}{\P p_z^b} \\ +(t\,' - \frac{\P t}{\P s})\frac{\P f}{\P t} + (-d)'\frac{\P f}{\P(-d)}] \end{pmatrix} dV_\perp
$$

.

Finally, noting that $d' = 0$ and $t\,' = \frac{\P t}{\P s} - D_x{}'\,p_x^b + D_x{}''\,x^b = \frac{\P t}{\P s} + O(e)$, we integrate the transverse derivatives by parts, also noting that for Hamiltonian motion $\frac{\P(x^b)'}{\P x^b} + \frac{\P(p_x^b)'}{\P p_x^b} = \frac{\P(z^b)'}{\P z^b} + \frac{\P(p_z^b)'}{\P p_z^b} = 0$, and average to obtain

$$\frac{d}{ds}[\lambda\langle g(s,\tau,\delta)\rangle] = \lambda\left\langle \begin{array}{l} (x^\beta)'\frac{\partial g}{\partial x^\beta} + (p_x^\beta)'\frac{\partial g}{\partial p_x^\beta} + (z^\beta)'\frac{\partial g}{\partial z^\beta} \\ +(p_z^\beta)'\frac{\partial g}{\partial p_z^\beta} + \tau'\frac{\partial g}{\partial \tau} + \frac{\partial g}{\partial s} \end{array} \right\rangle + O(\varepsilon)\,. \tag{19}$$

By choosing $g = 1$ we obtain the result that $\frac{d\lambda}{ds} = O(\varepsilon)$, so that to lowest order

$$\frac{d}{ds}\langle g(s,\tau,\delta)\rangle = \left\langle (x^\beta)'\frac{\partial g}{\partial x^\beta} + (p_x^\beta)'\frac{\partial g}{\partial p_x^\beta} + (z^\beta)'\frac{\partial g}{\partial z^\beta} + (p_z^\beta)'\frac{\partial g}{\partial p_z^\beta} + \tau'\frac{\partial g}{\partial \tau} + \frac{\partial g}{\partial s}\right\rangle . \tag{20}$$

Before proceeding with the derivation of rms envelope equations, consider two functions $g(x^\beta, p_x^\beta, z^\beta, p_z^\beta, \tau, -\delta, s)$ and $h(x^\beta, p_x^\beta, z^\beta, p_z^\beta, \tau, -\delta, s)$. Defining variations $\Delta g = g - \langle g\rangle$ and $\Delta h = h - \langle h\rangle$, we find that

$$\langle \Delta g \Delta h\rangle = \langle gh\rangle - \langle g\rangle\langle h\rangle\,. \tag{21}$$

Setting $h = g$ leads to the standard deviation of g

$$\sigma_g^2 = \langle(\Delta g)^2\rangle = \langle g^2\rangle - \langle g\rangle^2\,. \tag{22}$$

Equations (20), (21), and (22) will be used extensively in obtaining the rms envelope equations.

THE RMS ENVELOPE EQUATIONS

Let $q = x^\beta$ or $q = z^\beta$ be one of the transverse canonical coordinates and let $p = q'$ be the momentum canonical to q. Then, according to Eq. (21), to lowest order in ε

$$
\begin{aligned}
\frac{d}{ds}\sigma_q^{\,2} &= \frac{d}{ds}\{\langle q^2\rangle - \langle q\rangle^2\} \\
&= 2\{\langle q'q\rangle - \langle q'\rangle\langle q\rangle\} \\
&= 2\{\langle pq\rangle - \langle p\rangle\langle q\rangle\} \\
&= 2\langle \Delta p\Delta q\rangle \\
\frac{d^2}{ds^2}\sigma_q^{\,2} &= \frac{d}{ds}\{\frac{d}{ds}\sigma_q^{\,2}\} \\
&= 2\frac{d}{ds}\{\langle pq\rangle - \langle p\rangle\langle q\rangle\} \\
&= 2\{\langle p^2\rangle - \langle p\rangle^2 + \langle qp'\rangle - \langle q\rangle\langle p'\rangle\} \\
&= 2\{\langle \sigma_p^{\,2}\rangle - \left\langle \Delta q\Delta\frac{\partial H}{\partial q}\right\rangle\} \\
\frac{d}{ds}\sigma_p^{\,2} &= \frac{d}{ds}\{\langle p^2\rangle - \langle p\rangle^2\} \\
&= 2\{\langle pp'\rangle - \langle p\rangle\langle p'\rangle\} \\
&= -2\{\left\langle p\frac{\partial H}{\partial q}\right\rangle - \langle p\rangle\left\langle \frac{\partial H}{\partial q}\right\rangle\} \\
&= -2\left\langle \Delta p\Delta\frac{\partial H}{\partial q}\right\rangle,
\end{aligned}
\tag{23}
$$

where $p' = -\dfrac{\partial H}{\partial q}$.

Let us now define an effective emittance in terms of the rms statistical emittance by

$$\varepsilon_q = \eta_e\{\sigma_p^{\,2}\sigma_q^{\,2} - \langle\Delta p\Delta q\rangle^2\}^{\frac{1}{2}}, \tag{24}$$

where the factor η_e is a constant normalization factor. For a K-V distribution the factor $\eta_e = 4$ makes the effective emittance equal to the area of the maximal (q,p) phase space ellipse. Using Eqs. (23) and (24), it is straightforward to show that

$$\frac{d}{ds}\left(\frac{\varepsilon_q}{\eta_e}\right)^2 = -2\{\langle(\Delta q)^2\rangle\left\langle \Delta p\Delta\frac{\partial H}{\partial q}\right\rangle - \left\langle \Delta q\Delta\frac{\partial H}{\partial q}\right\rangle\langle\Delta p\Delta q\rangle\}\,. \tag{25}$$

If the dependence of $\frac{¶H}{¶q}$ on q is linear, i.e. $\frac{¶H}{¶q} = \mathrm{K}_q q$, then Eq. (25) shows that the effective emittance is constant. For the Hamiltonian of Eq. (13), $\frac{¶H}{¶q} = \mathrm{K}_q q + O(e^2)$ for both x and z , so the effective emittance is constant to leading order.

The relations $\langle \Delta p \Delta q \rangle = \frac{1}{2}\frac{d}{ds} s_q^{\,2}$ and $\langle s_p^{\,2} \rangle = \frac{1}{2}\frac{d^2}{ds^2} s_q^{\,2} + \left\langle \Delta q \Delta \frac{¶H}{¶q} \right\rangle$, Eq. (23), can be combined with the definition, Eq. (25), of the effective emittance to obtain, using the identity $(s_q^{\,2})'' = \frac{[(s_q^{\,2})']^2}{2 s_q^{\,2}} + 2 s_q s_x''$, the rms envelope equation for q :

$$(h_e^{\frac{1}{2}} s_q)'' - \frac{e_q^{\,2}}{(h_e^{\frac{1}{2}} s_q)^3} + \frac{h_e}{(h_e^{\frac{1}{2}} s_q)} \left\langle \Delta q \Delta \frac{¶H}{¶q} \right\rangle = 0 \,. \tag{26}$$

Equation (26) shows that the normalization constant h_e provides a scale factor relating the rms value of Δq to an envelope radius $h_e^{\frac{1}{2}} s_q$.

Using the adopted beam Hamiltonian, we obtain

$$\begin{aligned} a'' - \frac{e_x^{\,2}}{a^3} + [K_x(s) + \frac{q\Phi^{20}(s)}{g^2 b^2 E_0}]a = 0 \\ b'' - \frac{e_x^{\,2}}{b^3} + [K_y(s) + \frac{q\Phi^{02}(s)}{g^2 b^2 E_0}]b = 0, \end{aligned} \tag{27}$$

where $a = h_e^{\frac{1}{2}} s_x$ and $b = h_e^{\frac{1}{2}} s_z$. Equation (27) presents the rms envelope equations, correct to lowest order in e , for betatron oscillations at momentum d . These equations are independent of dispersion and momentum. For a monoenergetic beam, no momentum spread, and an elliptical K-V distribution of radii a and b , the potential is given by $\Phi_{K-V} = -\frac{1}{2}[\frac{4q\, l_{tot}}{a(a+b)}]x^2 - \frac{1}{2}[\frac{4q\, l_{tot}}{b(a+b)}]z^2$, and Eq. (27) takes the usual form

$$\begin{aligned} a'' - \frac{e_x^{\,2}}{a^3} + K_x(s)a = [\frac{4q^2 l_{tot}}{g^2 b^2 E_0}]\frac{1}{a+b} = \frac{2K}{a+b} \\ b'' - \frac{e_x^{\,2}}{b^3} + K_y(s)b = [\frac{4q^2 l_{tot}}{g^2 b^2 E_0}]\frac{1}{a+b} = \frac{2K}{a+b}, \end{aligned} \tag{28}$$

where $K = \dfrac{2q^2 l_{tot}}{g^2 b^2 E_0}$ is the generalized perveance (7).

THE SPACE CHARGE DISTRIBUTION

The rms envelope equations derived above provide only part of the formulation of a particle core model for rings. It remains to obtain the space charge forces to close the model. This can be done by superposing the particle distributions at each energy, which can be inferred from the envelope equations, over the momentum distribution, while taking into account the effects of dispersion in spreading the beam. To do this it is necessary to make some assumptions about the particle distribution function. This is a standard problem when dealing with moments expansions, and we state below where this is required.

For a given momentum, d, the overall displacement relative to the center of the reference beam is seen from Eq. (7) to be

$$\begin{aligned} \tilde{x} &= x^b + D_x d \\ \tilde{z} &= z^b \,. \end{aligned} \tag{29}$$

The total particle density at each value of $\tilde{x}$ and $\tilde{z}$ is given by the distribution function as follows

$$\begin{aligned} n(\tilde{x},\tilde{z}) &= \int \begin{pmatrix} f(x^b, p_x^b, z^b, p_z^b, t, -d, s)\, d(t - bct - D_x p_x^b + D_x' x^b) \\ \times d(x^b + D_x d - \tilde{x})\, d(z^b - \tilde{z})\, dV \end{pmatrix} \\ &= \int f(\tilde{x} - D_x d, p_x^b, \tilde{z}, p_z^b, bct + D_x p_x^b - D_x' x^b, -d, s)\, dp_x^b\, dp_z^b\, d(-d) \\ &= \int n(\tilde{x},\tilde{z},-d)\, d(-d). \end{aligned} \tag{30}$$

Equation (30) requires detailed knowledge of the distribution function, but it is of interest to extend the moments approach of the previous sections to obtain rms values of the overall charge distribution.

We now calculate the overall average and rms values of $\tilde{x}$ and $\tilde{z}$:

$$\begin{aligned}
\left\langle \tilde{x} \right\rangle_{tot} &= \int (\langle x \rangle + D_x d) P(d) d(-d) \\
&= \langle x \rangle_{tot} + D_x \langle d \rangle_{tot} \\
\left\langle \tilde{z} \right\rangle_{tot} &= \int \langle z \rangle P(d) d(-d) \\
&= \langle z \rangle_{tot} .
\end{aligned} \tag{31}$$

If the particle distribution at each energy is centered about the closed orbit, then $\langle x \rangle = \langle z \rangle = \langle x \rangle_{tot} = \langle z \rangle_{tot} = 0$; and if the reference orbit is centered in the momentum distribution, then $\langle d \rangle_{tot} = 0$. In this case, $\left\langle \tilde{x} \right\rangle_{tot} = \left\langle \tilde{z} \right\rangle_{tot} = 0$. To evaluate the total rms size of the beam, we calculate

$$\begin{aligned}
\left\langle (\Delta \tilde{x})^2 \right\rangle_{tot} &= \left\langle [(x^b + D_x d) - (\langle x^b \rangle_{tot} + D_x \langle d \rangle_{tot})]^2 \right\rangle_{tot} \\
&= \left\langle \begin{matrix} (x^b - \langle x^b \rangle_{tot})^2 \\ +2D_x (x^b - \langle x^b \rangle_{tot})(d - \langle d \rangle_{tot}) + D_x{}^2 (d - \langle d \rangle_{tot})^2 \end{matrix} \right\rangle_{tot} \\
&= [\langle (x^b)^2 \rangle_{tot} - (\langle x^b \rangle_{tot})^2] \\
&\quad +2D_x [\langle x^b d \rangle_{tot} - \langle x^b \rangle_{tot} \langle d \rangle_{tot}] + D_x{}^2 [\langle d^2 \rangle_{tot} - (\langle d \rangle_{tot})^2] \\
\left\langle (\Delta \tilde{z})^2 \right\rangle_{tot} &= \left\langle (z - \langle z \rangle_{tot})^2 \right\rangle_{tot} \\
&= \langle z^2 \rangle_{tot} - (\langle z \rangle_{tot})^2 .
\end{aligned} \tag{32}$$

If $\langle x \rangle_{tot} = \langle z \rangle_{tot} = 0$, if the betatron oscillation coordinates are uncorrelated with d, so that $\langle xd \rangle_{tot} = 0$, and if $\langle d \rangle_{tot} = 0$, then the effective emittances are independent of momentum, and Eq. (32) simplifies to

$$\begin{aligned}
s_{\tilde{x}}{}^2 &= \left\langle (\Delta \tilde{x})^2 \right\rangle_{tot} & s_{\tilde{z}}{}^2 &= \left\langle (\Delta \tilde{z})^2 \right\rangle_{tot} \\
&= \langle x^2 \rangle_{tot} + D_x^2 \langle d^2 \rangle_{tot} & &= \langle z^2 \rangle_{tot} \\
&= \frac{a^2}{4} + D_x^2 \langle d^2 \rangle_{tot} & &= \frac{b^2}{4} .
\end{aligned} \tag{33}$$

Consequently the rms standard deviations of the overall displacement contain separate contributions from betatron oscillations and from dispersion in the case of $\tilde{x}$ and from betatron oscillations only in the case of $\tilde{z}$.

The above results can be used to construct simple overall particle distributions for the purpose of calculating the space charge fields for particle core model dynamics and particle tracking. The use of Eq. (30) to define the particle density requires detailed information about the distribution function. A simpler computational approach involves assuming some standard form for the distribution function in transverse phase space. Then the envelope equations can be solved to provide the rms radii of the assumed distribution, and hence its contribution to the space charge density, at each energy. Discretizing momentum space into a number of bins, each of size Δd, and assuming a fraction of the beam, $P(d)\Delta d$, to be inside each bin, the associated charge distributions can be added in a weighted sum with weights proportional to $P(d)\Delta d$. The total space charge field is thus obtained by superposition of the fields given by the transverse distribution parameterized by the envelope solutions. For self-consistency, the total space charge potential obtained in this manner, expanded about the reference closed orbit, must yield the coefficients used in Eqs. (10) and (13). To enforce this, iteration of the process may be required.

Even more simply, the overall rms beam parameters in Eq. (33) can be associated with an assumed transverse distribution, and the summation over beam momentum bypassed. For example, for an elliptical K-V distribution with semi-axes

$$\begin{aligned} A &= 2s_{\tilde{x}} & B &= 2s_{\tilde{z}} \\ &= \left(a^2 + 4D_x^2 \left\langle d^2 \right\rangle_{tot}\right)^{\frac{1}{2}} & &= b, \end{aligned} \tag{34}$$

the space charge potential is

$$\begin{aligned} \Phi_{K-V} &= -\frac{1}{2}[\frac{4ql_{tot}}{A(A+B)}]\tilde{x}^2 - \frac{1}{2}[\frac{4ql_{tot}}{B(A+B)}]\tilde{z}^2 \\ &= -\frac{1}{2}[\frac{4ql_{tot}}{A(A+B)}](x^b + D_x d)^2 - \frac{1}{2}[\frac{4ql_{tot}}{B(A+B)}](z^b)^2, \end{aligned} \tag{35}$$

so that $\Phi^{20} = -\dfrac{4ql_{tot}}{A(A+B)}$ and $\Phi^{02} = -\dfrac{4ql_{tot}}{B(A+B)}$. With this approximation, the envelope equations must be solved in the form of Eq. (27), rather than Eq. (28). Also, through the space charge terms, the radii A and B appear in Eq. (12) for the dispersion, D_x. An iterative approach to solving for D_x for a matched beam prior to the dynamic calculations can be developed starting with the usual dispersion function without space charge. The approach requires solving Eqs. (27), (34), and (35) simultaneously for matched core boundary conditions, using the available dispersion function. With an explicit symplectic integration scheme of independent kicks and linear transport steps, all information would be available when required. After solving for the matched envelope, the resulting space charge potential could be used to solve for the space-charge-corrected closed orbit dispersion function in Eq. (12). This would complete one iteration, and the approach could be repeated to convergence. The converged space-charge-corrected dispersion function for a matched beam would then be used as the dispersion function for the dynamic calculations. This simple scheme provides a self-consistent particle core model using a space-charge-corrected dispersion function, a pair of rms envelope equations, and K-V beam distribution.

CONCLUSIONS

A self-consistent particle core model for transverse beam dynamics in rings, including the effects of space charge and dispersion, was derived using a moments approach. The model includes rms envelope equations for betatron oscillations, a space-charge-corrected dispersion function, and a prescription for the evaluation of the space charge potential, all coupled together self-consistently. In addition to describing the collective dynamics of the beam, this model can provide space charge forces for particle tracking calculations. The derivation was carried out using an accelerator ordering scheme and a statistical moments analysis based on the canonical distribution function in the six-dimensional phase space of the beam Hamiltonian. The azimuthal variation of the second moments of the transverse canonical (betatron oscillation) coordinates, x^b and z^b, averaged at fixed beam energy over the kinetic distribution function of the beam, leads to the rms envelope equations. These envelope equations are found to be independent of dispersion. A subsequent integration over beam energies provides the rms values of the overall displacements, $\tilde{x}$ and $\tilde{z}$, and the spatial beam distribution and space charge force. This integration can be carried out with varying degrees of complexity to obtain particular models of the charge distribution. Self-consistency must be assured by using the resulting space charge force to calculate the dispersion and envelope equations until the system is converged, thus closing the loop.

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